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Research Article

Dynamics of a Rational System of Difference Equations in the Plane

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We consider a rational system of first-order difference equations in the plane with four parameters such that all fractions have a common denominator. We study, for the different values of the parameters, the global and local properties of the system. In particular, we discuss the boundedness and the asymptotic behavior of the solutions, the existence of periodic solutions, and the stability of equilibria.

1. Introduction

In recent years, rational difference equations have attracted the attention of many researchers for varied reasons. On the one hand, they provide examples of nonlinear equations which are, in some cases, treatable but whose dynamics present some new features with respect to the linear case. On the other hand, rational equations frequently appear in some biological models, and, hence, their study is of interest also due to their applications. A good example of both facts is Ricatti difference equations; the richness of the dynamics of Ricatti equations is very well-known (see, e.g., [1, 2]), and a particular case of these equations provides the classical Beverton-Holt model on the dynamics of exploited fish populations [3]. Obviously, higher-order rational difference equations and systems of rational equations have also been widely studied but still have many aspects to be investigated. The reader can find in the following books [4–6], and the works cited therein, many results, applications, and open problems on higher-order equations and rational systems.

A preliminar study of planar rational systems in the large can be found in the paper [7] by Camouzis et al. In such work, they give some results and provide some open questions for systems of equations of the type

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters are taken to be nonnegative. As shown in the cited paper, some of those systems can be reduced to some Ricatti equations or to some previously studied second-order rational equations. Further, since, for some choices of the parameters, one obtains a system which is equivalent to the case with some other parameters, Camouzis et al. arrived at a list of 325 nonequivalent systems to which the attention should be focused. They list such systems as pairs (k, l) where k and l make reference to the number of the corresponding equation in their Tables 3 and 4.

In this paper, we deal with the rational system labelled ((21) and (23)) in [7]. Note that, for nonnegative coefficients, such a system is neither cooperative nor competitive, but it has the particularity that denominators in both equations are equal. This allows us to use some of the techniques developed in [8] to completely obtain the solutions and give a nice description of the dynamics of the system. In principle, we will not restrict ourselves to the case of nonnegative parameters, although this case will be considered in detail in the last section. Hence, we will study the general case of the system

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n}{y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n}{y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given real numbers, and the initial condition (x_0, y_0) is an arbitrary vector of $\mathbb{R}^2$. It should be noticed that when $\alpha_1\beta_2 = \alpha_2\beta_1$ the system can be reduced to a Ricatti equation (or it does not admit any complete solution, which occurs for $\alpha_2 = \beta_2 = 0$) and therefore these cases will be neglected. Since we will not assume nonnegativeness for neither the coefficients nor the initial conditions, a forbidden set will appear. We will give an explicit characterization of the forbidden set in each case. Obviously, all the results concerning solutions that we will state in the paper are to be applied only to complete orbits. We will focus our attention on three aspects of the dynamics of the system: the boundedness character and asymptotic behavior of its solutions, the existence of periodic orbits (in particular, of prime period-two solutions), and the stability of the equilibrium points. It should be remarked that, depending on the parameters, they may appear asymptotically stable fixed points, stable but not asymptotically stable fixed points, nonattracting unstable fixed points, and attracting unstable fixed points.

The paper is organized, besides this introduction, in three sections. Section 2 is devoted to some preliminaries and some results which can be mainly deduced from the general situation studied in [8]. Next, we study the case $\beta_2 = 0$ since such assumption yields the uncoupled globally 2-periodic equation $y_{n+1} = \alpha_2/y_n$ and the system is reduced to a linear first-order equation with 2-periodic coefficients; this will be our Section 3. The main section of the paper is Section 4, where we give the solutions to the system and the description of the dynamics in the general case $\beta_2 \neq 0$. We finish the paper by describing the dynamics in the particular case where the coefficients and the initial conditions are taken to be nonnegative.

2. Preliminaries and First Results

Systems of linear fractional difference equations $X_{n+1} = F(X_n)$ in which denominators are common for all the components of F have been studied in [8]. If one denotes by q the mapping given by $q(a_1, a_2, \ldots, a_{k+1}) = (a_1/a_{k+1}, a_2/a_{k+1}, \ldots, a_k/a_{k+1})$ for $(a_1, a_2, \ldots, a_{k+1}) \in \mathbb{R}^{k+1}$ with $a_{k+1} \neq 0$ and $\ell : \mathbb{R}^k \to \mathbb{R}^{k+1}$ is given by $\ell(a_1, a_2, \ldots, a_k) = (a_1, a_2, \ldots, a_k, 1)$, it is shown in such work that the system can be written in the form $X_{n+1} = q \circ A \circ \ell(X_n)$, where A is a $(k+1) \times (k+1)$ square matrix constructed with the coefficients of the system. In the special case of our system (1.2) one actually has

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = q \circ \left(\begin{pmatrix} \beta_1 & 0 & \alpha_1 \\ \beta_2 & 0 & \alpha_2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ 1 \end{pmatrix} \right). \tag{2.1}$$

This form of the system lets us completely determine its solutions in terms of the powers of the associated matrix

$$A = \begin{pmatrix} \beta_1 & 0 & \alpha_1 \\ \beta_2 & 0 & \alpha_2 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.2}$$

Actually, the explicit solution to the system with initial condition (x_0, y_0) is given by

$$(x_{n+1}, y_{n+1})^t = q \circ A^n (x_0, y_0, 1)^t, \tag{2.3}$$

where M^t stands for the transposed of a matrix M. Therefore, our system can be completely solved, and the solution starting at (x_0, y_0) is just the projection by q of the solution of the linear system $\mathrm{X}_{n+1} = A\mathrm{X}_n$ with initial condition $\mathrm{X}_0 = (x_0, y_0, 1)^t$ whenever such projection exists.

Remark 2.1. When such projection does not exist, then (x_0, y_0) lies in the forbidden set. Clearly, this may only happen when, for some $n \geq 1$, one has

$$(0, 0, 1) A^n (x_0, y_0, 1)^t = 0. \tag{2.4}$$

Therefore, if $a_i(n) \in \mathbb{R}$, $0 \leq i \leq 2$ are such that $A^n = a_0(n) I + a_1(n) A + a_2(n) A^2$, then one immediately obtains that the forbidden set is given by the following union of lines:

$$\mathbf{F} = \bigcup_{n \geq 1} \left\{ (x_0, y_0) \in \mathbb{R}^2 : a_1(n) y_0 + a_2(n) \beta_2 x_0 + a_2(n) \alpha_2 + a_0(n) = 0 \right\}. \tag{2.5}$$

The explicit calculation of $a_i(n)$, $0 \le i \le 2$ for each $n \ge 3$ may be done in several ways. For instance, one has that $a_0(n) + a_1(n)x + a_2(n)x^2$ is the remainder of the division of x^n by the characteristic polynomial of A. Further, by elementary techniques of linear algebra one can also compute them in terms of the eigenvalues of A (an approach using the solutions to an associated linear difference equation may be seen in [9]).

Remark 2.2. As mentioned in the introduction, all through the paper we will consider that

$$\beta_2\alpha_1 \neq \beta_1\alpha_2, \tag{2.6}$$

(this is to say that the matrix A is nonsingular) since the cases with $\beta_2\alpha_1 = \beta_1\alpha_2$ may be reduced to a single Ricatti equation. Actually, if $\alpha_2 = \beta_2 = 0$, then the system does not admit any complete solution, whereas, for $\alpha_2 \neq 0$ or $\beta_2 \neq 0$, one has that there exists a constant C such that $\alpha_1 = C\alpha_2$ and $\beta_1 = C\beta_2$, and hence the first equation of the system may be substituted by $x_{n+1} = Cy_{n+1}$ and then the second one reduces to the Ricatti equation

$$y_{n+2} = \frac{\alpha_2 + \beta_2 C y_{n+1}}{y_{n+1}}, \quad n = 0, 1, \dots, \tag{2.7}$$

with initial condition $y_1 = (\alpha_2 + \beta_2 x_0)/y_0$.

Our main goal will be to give a description of the dynamics of the system in terms of the eigenvalues of the associated matrix A given in (2.2). We begin with the following result concerning 2-periodic solutions which is the particularization to our system of the analogous general result given in Theorem 3.1 and Remark 3.1 of [8].

Proposition 2.3. *Consider the system* (1.2) *with* $\alpha_1\beta_2 \neq \alpha_2\beta_1$. *One has the following:*

(1) *If* $\beta_2 \neq 0$, *then there are exactly as many equilibria as distinct real eigenvalues of the matrix* A. *More concretely, for each real eigenvalue* λ, *one gets the equilibrium* $((\lambda^2 - \alpha_2)/\beta_2, \lambda)$.

(2) *When* $\beta_2 = 0$, *one finds that:*

  (a) *if* $\alpha_2 < 0$, *then there are no fixed points,*

  (b) *if* $0 < \alpha_2 \neq \beta_1^2$, *then there are two fixed points at* $(\alpha_1/(\sqrt{\alpha_2} - \beta_1), \sqrt{\alpha_2})$ *and* $(-\alpha_1/(\sqrt{\alpha_2} + \beta_1), -\sqrt{\alpha_2})$,

  (c) *if* $\alpha_2 = \beta_1^2$ *and* $\alpha_1 \neq 0$, *then the only equilibrium point is* $(-\alpha_1/2\beta_1, -\beta_1)$,

  (d) *if* $\alpha_2 = \beta_1^2$ *and* $\alpha_1 = 0$, *then there is an isolated fixed point* $(0, -\beta_1)$ *and a whole line of equilibria* (x_0, β_1).

(3) *There exist periodic solutions of prime period 2 if and only if* $\alpha_1\beta_2 = 0$.

Proof. As stated in [8], a point $(a, b) \in \mathbb{R}^2$ is an equilibrium if and only if $(a, b, 1)$ is an eigenvector of the associated matrix A. When $\beta_2 \neq 0$, it is straightforward to prove that, for each real eigenvalue λ, the vector $((\lambda^2 - \alpha_2)/\beta_2, \lambda, 1)$ is an eigenvector. In the case $\beta_2 = 0$, the equilibrium points can be easily computed directly from the equations $\alpha_2 = y^2$, $\alpha_1 + \beta_1 x = xy$.

For the proof of affirmation (2.3), it suffices to bear in mind that, according to [8], the existence of prime period-two solutions is only possible when the associated matrix A has an eigenvalue λ such that $-\lambda$ is also an eigenvalue. Since A is a 3×3 square matrix, this obviously implies that the trace of A is also an eigenvalue. Hence, β_1 is an eigenvalue, but this is only possible if $\alpha_1\beta_2 = 0$. If $\alpha_1 = 0$, then the initial condition $(0, y_0)$ gives a prime period 2 solution whenever $y_0^2 \neq \alpha_2$ whereas, if $\alpha_1 \neq 0$ and $\beta_2 = 0$, a direct calculation shows that the solution with initial conditions $(0, -\beta_1)$ is periodic of prime period 2. □

We now study the stability of fixed points in some of the cases. Recall that a fixed point of our system (x^*, y^*) always verifies $y^* = \lambda$ for some real eigenvalue λ of the matrix A. We will say in such case that the fixed point (x^*, y^*) is *associated* to λ.

Proposition 2.4. *Consider the system* (1.2) *with* $\alpha_1\beta_2 \neq \alpha_2\beta_1$. *Let* $\rho(A)$ *be the spectral radius of the matrix* A *given in* (2.2), *and let* λ *be an eigenvalue of* A.

(1) *If* $|\lambda| < \rho(A)$, *then the associated equilibrium is unstable.*

(2) *If* $|\lambda| = \rho(A)$ *and all the eigenvalues of* A *whose modulus is* $\rho(A)$ *are simple, then the associated fixed point is stable. Further, if in this case* λ *is the unique eigenvalue whose modulus is* $\rho(A)$, *then it is asymptotically stable.*

Proof. The Jacobian matrix of the map $F(x, y) = ((\alpha_1 + \beta_1 x)/y, (\alpha_2 + \beta_2 x)/y)$ at a fixed point (x^*, y^*) is given by

$$\mathrm{DF}(x^*, y^*) = \begin{pmatrix} \dfrac{\beta_1}{y^*} & -x^*/y^* \\ \dfrac{\beta_2}{y^*} & -1 \end{pmatrix}. \tag{2.8}$$

Consider an eigenvalue λ of A, and let λ_2, λ_3 be the other (nonnecessarily different) eigenvalues of A. Let us show that the eigenvalues of the Jacobian matrix at a fixed point associated to λ are just λ_2/λ and λ_3/λ. The result is trivial when $\beta_2 = 0$ since the eigenvalues of A are β_1 and $\pm\sqrt{\alpha_2}$ and fixed points are always associated to one of the eigenvalues $\pm\sqrt{\alpha_2}$. If $\beta_2 \neq 0$, then $x^* = (\lambda^2 - \alpha_2)/\beta_2$ and $y^* = \lambda$ and, therefore, one obtains

$$\begin{aligned} \operatorname{trace}(\mathrm{DF}(x^*, y^*)) &= \frac{\beta_1 - \lambda}{\lambda} = \frac{\lambda_2 + \lambda_3}{\lambda} \\ \det(\mathrm{DF}(x^*, y^*)) &= \frac{-\beta_1\lambda + \lambda^2 - \alpha_2}{\lambda^2} = \frac{\det(A)}{\lambda^3} = \frac{\lambda_2\lambda_3}{\lambda^2}, \end{aligned} \tag{2.9}$$

showing that the eigenvalues of $\mathrm{DF}(x^*, y^*)$ are as claimed. Now, the first statement follows at once since, if $|\lambda| < \rho(A)$, then at least one of the eigenvalues of $\mathrm{DF}(x^*, y^*)$ lies outside the unit circle. Moreover, when $|\lambda| = \rho(A)$ and it is the unique eigenvalue with such property, then the eigenvalues of $\mathrm{DF}(x^*, y^*)$ are inside the (open) unit ball, and, hence, the equilibrium (x^*, y^*) is asymptotically stable, which proves the second part of (2.2).

For the proof of the first part of (2.2), let us recall that if (x^*, y^*) is a fixed point of (1.2) associated to the real eigenvalue λ, then $X^* = (x^*, y^*, 1)^t$ is a fixed point of the linear system $X_{n+1} = (1/\lambda)AX_n$. The eigenvalues of the matrix $M = (1/\lambda)A$ are obviously $1, \lambda_2/\lambda$ and λ_3/λ. Since the eigenvalues of A having modulus $\rho(A)$ are simple, so are the eigenvalues of M having modulus 1. Therefore, the fixed point X^* is stable [2, Theorem 4.13]. Now, the stability of (x^*, y^*) follows at once from (2.3) and the continuity of q in the semispace $z > 0$. □

3. Case $\beta_2 = 0$

Recall that, since we are assuming that inequality (2.6) holds, we have $\beta_1\alpha_2 \neq 0$. In this case, the forbidden set of the system reduces to the line $y = 0$. Since $\beta_2 = 0$, the second equation of the system becomes the uncoupled equation

$$y_{n+1} = \frac{\alpha_2}{y_n}, \tag{3.1}$$

which, as far as $\alpha_2 \neq 0$, for each initial condition $y_0 \neq 0$ gives

$$y_n = \begin{cases} y_0 & \text{for even } n, \\ \dfrac{\alpha_2}{y_0} & \text{for odd } n. \end{cases} \tag{3.2}$$

Substituting such values in the first equation of the system, we obtain a first-order linear difference equation with 2-periodic coefficients whose solution is given by $x_1 = (\alpha_1 + \beta_1 x_0)/y_0$ and, for $n > 1$,

$$x_n = \begin{cases} \left(\dfrac{\beta_1^2}{\alpha_2}\right)^{n/2}\left[x_0 + \dfrac{\alpha_1(\beta_1 + y_0)}{\alpha_2}\displaystyle\sum_{k=1}^{n/2}\left(\dfrac{\alpha_2}{\beta_1^2}\right)^k\right] & \text{for even } n, \\ \dfrac{\alpha_1}{y_0} + \dfrac{\beta_1}{y_0}\left(\dfrac{\beta_1^2}{\alpha_2}\right)^{(n-1)/2}\left[x_0 + \dfrac{\alpha_1(\beta_1 + y_0)}{\alpha_2}\displaystyle\sum_{k=1}^{(n-1)/2}\left(\dfrac{\alpha_2}{\beta_1^2}\right)^k\right] & \text{for odd } n. \end{cases} \tag{3.3}$$

Hence, we have proved the following.

Proposition 3.1. *If $\beta_2 = 0$ and $\beta_2\alpha_1 \neq \beta_1\alpha_2$, then the system* (1.2) *is solvable for any initial condition (x_0, y_0) with $y_0 \neq 0$ and the solution (x_n, y_n) is given by* (3.2) *and* (3.3) *where, explicitly, one finds the following:*

(1) *If $\alpha_2 = \beta_1^2$, then for $n > 1$*

$$x_n = \begin{cases} x_0 - \dfrac{\alpha_1(\beta_1 + y_0)n}{2\beta_1^2} & \textit{for even } n, \\ \dfrac{\alpha_1}{y_0} + \dfrac{\beta_1 x_0}{y_0} - \dfrac{\alpha_1(\beta_1 + y_0)(n-1)}{2\beta_1 y_0} & \textit{for odd } n. \end{cases} \tag{3.4}$$

(2) *If $\alpha_2 \neq \beta_1^2$, then for $n > 1$*

$$x_n = \begin{cases} \left(\dfrac{\beta_1^2}{\alpha_2}\right)^{n/2}\left[x_0 + \dfrac{\alpha_1(\beta_1 + y_0)}{\beta_1^2 - \alpha_2}\left(1 - \left(\dfrac{\alpha_2}{\beta_1^2}\right)^{n/2}\right)\right] & \textit{for even } n, \\ \dfrac{\alpha_1}{y_0} + \dfrac{\beta_1}{y_0}\left(\dfrac{\beta_1^2}{\alpha_2}\right)^{(n-1)/2}\left[x_0 + \dfrac{\alpha_1(\beta_1 + y_0)}{\beta_1^2 - \alpha_2}\left(1 - \left(\dfrac{\alpha_2}{\beta_1^2}\right)^{(n-1)/2}\right)\right] & \textit{for odd } n. \end{cases} \tag{3.5}$$

From the proposition above, one can easily derive the following result which completely describes the asymptotic behaviour of the solutions to the system.

Corollary 3.2. *Consider $\beta_2 = 0$ and $\beta_1\alpha_2 \neq 0$.*

(1) *When $\beta_1^2 = \alpha_2$ one finds that*

 (a) *if $\alpha_1 \neq 0$, then every solution to the system is unbounded except those with initial condition $(x_0, -\beta_1)$, which are 2-periodic,*

 (b) *if $\alpha_1 = 0$, the system is globally 2-periodic.*

(2) *If $\beta_1^2 = -\alpha_2$, then the system (1.2) is globally 4-periodic. Further, the solution corresponding with the initial condition (x_0, y_0) is of prime period 2 if and only if $2\beta_1^2 x_0 + \alpha_1(\beta_1 + y_0) = 0$.*

(3) *If $\beta_1^2 \neq |\alpha_2|$, then the solutions with initial condition $((\alpha_1(\beta_1+y_0))/(\alpha_2-\beta_1^2), y_0)$ are period-two solutions. Moreover,*

 (a) *if $\beta_1^2 > |\alpha_2|$, then any other solution to the system (1.2) is unbounded,*

 (b) *if $\beta_1^2 < |\alpha_2|$, then any other solution of (1.2) is bounded and tends to one of the period-two solutions described above.*

Proof. The proof is a straightforward consequence of the explicit formulas for x_n and y_n given in Proposition 3.1. It should, however, be mentioned that the globally periodicity of the system in the case $\beta_1^2 = -\alpha_2$ can be easily seen since the associated matrix A given by (2.2) in such case verifies $A^4 = \beta_1^4 I$, where I stands for the identity matrix. Actually, a simple calculation proves that the solution starting at (x_0, y_0) is the 4-cycle

$$\left\{(x_0, y_0), \left(\frac{\alpha_1 + \beta_1 x_0}{y_0}, \frac{-\beta_1^2}{y_0}\right), \left(-x_0 - \frac{\alpha_1(\beta_1 + y_0)}{\beta_1^2}, y_0\right), \left(\frac{-\beta_1^2 x_0 + \alpha_1 y_0}{\beta_1 y_0}, \frac{-\beta_1^2}{y_0}\right)\right\}, \tag{3.6}$$

which is obviously 2-periodic if and only if $x_0 = -x_0 - (\alpha_1(\beta_1 + y_0))/\beta_1^2$. □

From the above result and Proposition 2.4, one easily get the following information about the stability of the fixed points.

Corollary 3.3. *Consider $\beta_2 = 0$ and $\beta_1\alpha_2 \neq 0$.*

(1) *If $\beta_1^2 = \alpha_2$, then*

 (a) *for $\alpha_1 \neq 0$, the unique fixed point of (1.2) is unstable,*

 (b) *for $\alpha_1 = 0$, every fixed point of (1.2) is stable but not asymptotically stable.*

(2) *If $\beta_1^2 \neq \alpha_2 > 0$, then*

 (a) *for $\beta_1^2 > \alpha_2$, both fixed points of (1.2) are unstable,*

 (b) *for $\beta_1^2 < \alpha_2$, the fixed points of (1.2) are stable but not asymptotically stable.*

4. Case $\beta_2 \neq 0$

Proposition 4.1. *Suppose $\beta_2 \neq 0$ and (x_0, y_0) is an initial condition not belonging to the forbidden set* **F**. *In such case, the solution of system* (1.2) *is given by*

$$x_n = \frac{v_{n+1}}{v_{n-1}} \frac{1}{\beta_2} - \frac{\alpha_2}{\beta_2}, \qquad y_n = \frac{v_n}{v_{n-1}}, \tag{4.1}$$

where v_n is the unique solution of the linear difference equation

$$v_{n+3} - \beta_1 v_{n+2} - \alpha_2 v_{n+1} + (\beta_1\alpha_2 - \beta_2\alpha_1) v_n = 0, \tag{4.2}$$

with initial conditions $v_{-1} = 1$, $v_0 = y_0$, and $v_1 = \beta_2 x_0 + \alpha_2$.

Proof. As we have seen in Section 2, the solution to System (1.2) starting at a point (x_0, y_0) not belonging to the forbidden set is just the projection by q of the solution of the linear system $(u_{n+1}, v_{n+1}, w_{n+1})^t = A(u_n, v_n, w_n)^t$ with initial condition $(x_0, y_0, 1)^t$, where A is given by (2.2). Since the third equation of such linear systems reads $w_{n+1} = v_n$, it can be reduced to the planar linear system of second-order equations

$$\begin{aligned} u_{n+1} &= \beta_1 u_n + \alpha_1 v_{n-1}, \\ v_{n+1} &= \beta_2 u_n + \alpha_2 v_{n-1}, \end{aligned} \tag{4.3}$$

and hence, if (u_n, v_n) is the solution to (4.3) obtained for the initial conditions $(u_0, v_0, v_{-1}) = (x_0, y_0, 1)$, then the solution of our rational system for the initial values (x_0, y_0) will be

$$x_{n+1} = \frac{u_n}{v_{n-1}}, \qquad y_{n+1} = \frac{v_n}{v_{n-1}}. \tag{4.4}$$

It is clear that for $\beta_2 \neq 0$, we have that u_n can be completely determined by (4.3) in terms of v_{n+1} and v_{n-1}, and hence it suffices to solve the third-order linear equation

$$v_{n+3} - \beta_1 v_{n+2} - \alpha_2 v_{n+1} + (\beta_1\alpha_2 - \beta_2\alpha_1) v_n = 0 \tag{4.5}$$

trivially deduced from (4.3) and substitute the corresponding values in (4.4) to obtain the result claimed. □

In the following results, we will discuss the behavior of the solutions to (1.2) by using Proposition 4.1. We shall consider three different cases depending on the roots of the characteristic polynomial of the linear equation (4.2). Recall that such roots are also the (possibly complex) eigenvalues of the matrix A given in (2.2).

From Proposition 4.1, we see that the asymptotic behavior of the solutions of System (1.2) will depend on the asymptotic behavior of the sequences $v_n/(v_{n-1})$, v_n being solutions of the linear difference equation (4.2). The theorem of Poincaré [2, Theorem 8.9] establishes a general result for the existence of $\lim_{n\to\infty} v_n/(v_{n-1})$. In our case, since (4.2) has constant coefficients, we can directly do the calculations, even in the cases not covered by the Theorem of Poincaré, to describe the dynamics of system (1.2).

4.1. The Characteristic Polynomial Has No Distinct Roots with the Same Module

Let λ_1, λ_2, and λ_3 be the three roots of the characteristic polynomial of the linear difference equation (4.2) in this case. A condition on the coefficients for this case can be given by

$$\left(\frac{(2/3)\beta_1\alpha_2 - \beta_2\alpha_1 - (2/27)\beta_1^3}{2}\right)^2 \leq \left(\frac{\alpha_2 + (1/3)\beta_1^2}{3}\right)^3, \tag{4.6}$$

with $\alpha_1 \neq 0$ or $\alpha_2 \leq 0$. Recall that we assume here that $\beta_2\alpha_1 \neq \beta_1\alpha_2$ and $\beta_2 \neq 0$.

If λ_1 is the characteristic root of maximal modulus, we will denote by L the line

$$L = \{(x, y) : \beta_2 x = (\beta_1 - \lambda_1)(y + \lambda_1)\}. \tag{4.7}$$

Proposition 4.2. *Suppose that $\beta_2 \neq 0$ and every root of the characteristic polynomial of the linear difference equation* (4.2) *is real and no two distinct roots have the same module. When (x_0, y_0) is not in the forbidden set, one finds the following:*

(1) *If $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then*

(a) *System* (1.2) *admits exactly the three equilibria $((\lambda_i^2 - \alpha_2)/\beta_2, \lambda_i)$, $i = 1, 2, 3$,*

(b) *the fixed point $((\lambda_1^2 - \alpha_2)/\beta_2, \lambda_1)$ attracts every complete solution starting on a point (x_0, y_0) which does not belong to the line L,*

(c) *the corresponding solution to the system with initial condition $(x_0, y_0) \neq ((\lambda_3^2 - \alpha_2)/\beta_2, \lambda_3)$ and $(x_0, y_0) \in L$ converges to $((\lambda_2^2 - \alpha_2)/\beta_2, \lambda_2)$.*

(2) *If $|\lambda_1| > |\lambda_2|$ and λ_1 has algebraic multiplicity 2, then*

(a) *System* (1.2) *admits exactly the two equilibria $((\lambda_i^2 - \alpha_2)/\beta_2, \lambda_i)$, $i = 1, 2$,*

(b) *the fixed point $((\lambda_1^2 - \alpha_2)/\beta_2, \lambda_1)$ attracts every complete solution except the other fixed point.*

(3) *If $|\lambda_1| > |\lambda_2|$ and λ_2 has algebraic multiplicity 2, then*

(a) *System* (1.2) *admits exactly the two equilibria $((\lambda_i^2 - \alpha_2)/\beta_2, \lambda_i)$, $i = 1, 2$,*

(b) *the fixed point $((\lambda_1^2 - \alpha_2)/\beta_2, \lambda_1)$ attracts every complete solution starting on a point (x_0, y_0) which does not belong to the line L,*

(c) *the corresponding solution to the system with initial condition $(x_0, y_0) \in L$ converges to $((\lambda_2^2 - \alpha_2)/\beta_2, \lambda_2)$.*

(4) *If λ_1 has multiplicity 3, then*

(a) *System* (1.2) *has a unique equilibrium $((\lambda_1^2 - \alpha_2)/\beta_2, \lambda_1)$,*

(b) *the equilibrium is a global attractor.*

Proof. In all the cases, the equilibrium points are directly given by Proposition 2.3. The assertions concerning the asymptotic behaviour can be derived as a consequence of Case 1 in [2, page 240], bearing in mind that

$$x_n = \frac{v_{n+1}}{v_{n-1}} \frac{1}{\beta_2} - \frac{\alpha_2}{\beta_2}, \qquad y_n = \frac{v_n}{v_{n-1}}, \tag{4.8}$$

and that v_n is the solution to the linear equation (4.2) with initial conditions $v_{-1} = 1$, $v_0 = y_0$, and $v_1 = \beta_2 x_0 + \alpha_2$. □

4.2. The Characteristic Polynomial Has Two Distinct Real Roots with the Same Module

It is easy to check that this case occurs when $\beta_1 \neq 0$, $\beta_2 \neq 0$, $\alpha_1 = 0$ and $\alpha_2 > 0$. Thus, the roots of the characteristic polynomial of the linear difference equation (4.2) are β_1 and $\pm\sqrt{\alpha_2}$.

Proposition 4.3. *Suppose $\beta_1 \neq 0$, $\beta_2 \neq 0$, $\alpha_1 = 0$ and $\alpha_2 > 0$. Assume also that (x_0, y_0) is not in the forbidden set.*

(1) *If $\beta_1^2 = \alpha_2$, then*

  (a) *there are two equilibrium points $(0, \pm\beta_1)$,*

  (b) *the equilibrium point $(0, \beta_1)$ attracts every complete solution not starting on a point of the line $x = 0$,*

  (c) *the solutions starting on a point (x_0, y_0) of the line $x = 0$ are prime period-two solutions except the two equilibrium points $(0, \pm\beta_1)$.*

(2) *If $\beta_1^2 > \alpha_2$, then*

  (a) *there are three equilibrium points $((\beta_1^2 - \alpha_2)/\beta_2, \beta_1)$ and $(0, \pm\sqrt{\alpha_2})$,*

  (b) *the equilibrium point $((\beta_1^2 - \alpha_2)/\beta_2, \beta_1)$ attracts every complete solution not starting on a point of the line $x = 0$,*

  (c) *the solutions starting on a point (x_0, y_0) of the line $x = 0$ are prime period-two solutions except the two equilibrium points $(0, \pm\sqrt{\alpha_2})$,*

(3) *If $\beta_1^2 < \alpha_2$, then*

  (a) *there are three equilibrium points $((\beta_1^2 - \alpha_2)/\beta_2, \beta_1)$ and $(0, \pm\sqrt{\alpha_2})$,*

  (b) *the solutions starting on a point of the line $x = 0$ are prime period-two solutions except the two equilibrium points $(0, \pm\sqrt{\alpha_2})$,*

  (c) *the solutions starting on a point of the lines $\beta_2 x + ((\alpha_2 - \beta_1^2)/\beta_1) y = 0$ or $x = (\beta_1^2 - \alpha_2)/\beta_2$ are unbounded with the only exception of the fixed point $((\beta_1^2 - \alpha_2)/\beta_2, \beta_1)$,*

  (d) *the solutions starting on any other point (x_0, y_0) are bounded and each tends to one of the two-periodic solutions.*

Proof. In all cases, the affirmation (a) is a consequence of Proposition 2.3.

When $\beta_1^2 = \alpha_2$, the roots are β_1, with algebraic multiplicity two, and $-\beta_1$. By Proposition 4.1, we know that any solution of the system can be written as

$$\beta_2 x_n = \frac{(n+1)P_1 + P_2 + P_3(-1)^{n+1}}{(n-1)P_1 + P_2 + P_3(-1)^{n-1}}\beta_1^2 - \beta_1^2,$$
$$y_n = \frac{nP_1 + P_2 + P_3(-1)^n}{(n-1)P_1 + P_2 + P_3(-1)^{n-1}}\beta_1, \tag{4.9}$$

where P_1, P_2, and P_3 actually satisfy

$$P_1 + P_2 - P_3 = \frac{\beta_2 x_0 + \beta_1^2}{\beta_1}, \qquad P_2 + P_3 = y_0, \qquad -P_1 + P_2 - P_3 = \beta_1. \tag{4.10}$$

If $P_1 \neq 0$, then (x_n, y_n) obviously tends to $(0, \beta_1)$. From (4.10), we see that $P_1 = 0$ if and only if $x_0 = 0$ and, in such case, $x_n = 0$ and y_n takes alternatively the values $\mathrm{A}\beta_1$ and $A^{-1}\beta_1$ with $A = (P_2 + P_3)/(P_2 - P_3)$. Notice that $y_0 \neq 0$ guaranties $P_2 + P_3 \neq 0$ and, since $\beta_1 \neq 0$, we can not have $P_1 = 0$ and $P_2 - P_3 = 0$. This completes the proof of (1.2).

In the case $\beta_1^2 \neq \alpha_2$, by Proposition 4.1, we can write the general solution of the system as

$$\beta_2 x_n = \frac{P_1 + \left[P_2 + P_3(-1)^{n+1}\right]\left(\sqrt{\alpha_2}/\beta_1\right)^{n+1}}{P_1 + \left[P_2 + P_3(-1)^{n-1}\right]\left(\sqrt{\alpha_2}/\beta_1\right)^{n-1}}\beta_1^2 - \alpha_2,$$
$$y_n = \frac{P_1 + \left[P_2 + P_3(-1)^{n}\right]\left(\sqrt{\alpha_2}/\beta_1\right)^{n}}{P_1 + \left[P_2 + P_3(-1)^{n-1}\right]\left(\sqrt{\alpha_2}/\beta_1\right)^{n-1}}\beta_1, \tag{4.11}$$

where P_1, P_2, and P_3 satisfy

$$P_1\beta_1 + (P_2 - P_3)\sqrt{\alpha_2} = \beta_2 x_0 + \alpha_2,$$
$$P_1 + P_2 + P_3 = y_0, \tag{4.12}$$
$$P_1\beta_1^{-1} + (P_2 - P_3)\sqrt{\alpha_2^{-1}} = 1.$$

When $\beta_1^2 > \alpha_2$, one immediately gets the results of statement (2.2) with an argument similar to that of the previous case. Therefore, we will focus our attention on the case $\beta_1^2 < \alpha_2$. The condition $x_0 = 0$ is, according to (4.12), equivalent to $P_1 = 0$, and, in such case, one gets $x_n = 0$ and y_n takes alternatively the values $K\sqrt{\alpha_2}$ and $K^{-1}\sqrt{\alpha_2}$ with $K = (P_2+P_3)/(P_2-P_3) = y_0/\alpha_2$. Now, if $P_1 \neq 0$ and the initial conditions are taken such that $P_2 + P_3 \neq 0 \neq P_2 - P_3$, then (x_n, y_n) tends obviously to the 2-cycle $\{(0, K\sqrt{\alpha_2}), (0, K^{-1}\sqrt{\alpha_2})\}$ where $K = (P_2+P_3)/(P_2-P_3)$. On the contrary, if either $P_2 + P_3 = 0$ or $P_2 - P_3 = 0$ (and only one of both equalities holds), then both sequences x_n and y_n are unbounded. From System (4.12), one gets that $P_2 - P_3 = 0$ if and only if $x_0 = (\beta_1^2 - \alpha_2)/\beta_2$ and that $P_2 + P_3 = 0$ is equivalent to $\beta_2 x_0 + ((\alpha_2 - \beta_1^2)/\beta_1)y_0 = 0$. This shows the validity of (c). □

4.3. *The Characteristic Polynomial Has Complex Roots*

Now, we consider the case in which the characteristic polynomial of the linear difference equation has a couple of complex roots $\rho e^{\pm i\theta}$, with $\sin\theta > 0$. Let $\lambda \neq 0$ be the real root. It can be easily shown that

$$\beta_1 = \lambda + 2\rho\cos\theta, \qquad \alpha_2 = -\left(2\lambda\rho\cos\theta + \rho^2\right), \qquad \beta_2\alpha_1 = \lambda\rho^2 + \beta_1\alpha_2, \tag{4.13}$$

and that this situation occurs when

$$\left(\frac{(2/3)\beta_1\alpha_2 - \beta_2\alpha_1 - (2/27)\beta_1^3}{2}\right)^2 > \left(\frac{\alpha_2 + (1/3)\beta_1^2}{3}\right)^3. \tag{4.14}$$

By Proposition 2.3, we know that the unique equilibrium is $((\lambda^2 - \alpha_2)/\beta_2, \lambda)$. Denote by L the line

$$L = \{(x, y) : \beta_2 x = (\beta_1 - \lambda)(y + \lambda)\}. \tag{4.15}$$

Notice that $(\beta_1 - \lambda)(y + \lambda) = 2y\rho\cos\theta - \alpha_2 - \rho^2$. Also, observe that the equilibrium does not belong to L.

Theorem 4.4. *Suppose $\beta_2 \neq 0$ and the characteristic polynomial of the linear difference equation have complex roots and assume that (x_0, y_0) is not in the forbidden set.*

(1) *The solutions starting on the line L remain on it, and they are either all periodic or all unbounded.*

(2) *If $|\lambda| > \rho$, then the unique equilibrium attracts all the solutions not starting on L.*

(3) *If $|\lambda| < \rho$, then every nonfixed bounded subsequence of a solution accumulates on L.*

(4) *If $|\lambda| = \rho$, then every complete solution (neither starting on the fixed point nor on L) lies on a nondegenerate conic, which does not contain the equilibrium.*

Proof. Assume that (x_0, y_0) is not the fixed point. Using Proposition 4.1, we have

$$\alpha_2 + \beta_2 x_n = \frac{P\lambda^{n+1} + 2\rho^{n+1}\cos(a + (n+1)\theta)}{P\lambda^{n-1} + 2\rho^{n-1}\cos(a + (n-1)\theta)},$$
$$y_n = \frac{P\lambda^{n} + 2\rho^{n}\cos(a + n\theta)}{P\lambda^{n-1} + 2\rho^{n-1}\cos(a + (n-1)\theta)}, \tag{4.16}$$

where the constants $P \in \mathbb{R}$ and $a \in [0, 2\pi)$, together with $k \in \mathbb{R}^+$, are given by

$$\begin{pmatrix} \lambda & \rho e^{i\theta} & \rho e^{-i\theta} \\ 1 & 1 & 1 \\ \dfrac{1}{\lambda} & \dfrac{e^{-i\theta}}{\rho} & \dfrac{e^{i\theta}}{\rho} \end{pmatrix} \begin{pmatrix} kP \\ ke^{ia} \\ ke^{-ia} \end{pmatrix} = \begin{pmatrix} \alpha_2 + \beta_2 x_0 \\ y_0 \\ 1 \end{pmatrix}. \tag{4.17}$$

Observe that we may consider $P \geq 0$, by replacing, if necessary, a with $a + \pi$.

Let us consider the sequences

$$\sigma_n = 2\left(\frac{\rho}{\lambda}\right)^n \cos(a + n\theta), \qquad \tau_n = 2\left(\frac{\rho}{\lambda}\right)^n \sin(a + n\theta). \tag{4.18}$$

It can be easily proved that

$$\alpha_2 + \beta_2 x_n = \lambda^2 \frac{P + \sigma_{n+1}}{P + \sigma_{n-1}}, \qquad y_n = \lambda \frac{P + \sigma_n}{P + \sigma_{n-1}}, \tag{4.19}$$

$$\lambda\sigma_{n+1} = \rho\sigma_n \cos\theta - \rho\tau_n \sin\theta, \qquad \rho\sigma_{n-1} = \lambda\sigma_n \cos\theta + \lambda\tau_n \sin\theta. \tag{4.20}$$

As a consequence, $\lambda^2\sigma_{n+1} - 2\lambda\rho\sigma_n\cos\theta + \rho^2\sigma_{n-1} = 0$, and then

$$\alpha_2 + \beta_2 x_n = 2\rho y_n \cos\theta - \rho^2 + P\frac{\lambda^2 - 2\rho\lambda\cos\theta + \rho^2}{P + \sigma_{n-1}}, \tag{4.21}$$

which is equivalent to

$$\beta_2 x_n - (\beta_1 - \lambda)(y_n + \lambda) = P\frac{\lambda^2 - 2\rho\lambda\cos\theta + \rho^2}{P + \sigma_{n-1}}. \tag{4.22}$$

Using (4.17), one has that $(x_0, y_0) \in L$ if and only if $P = 0$, and, from (4.22), we then get that $(x_n, y_n) \in L$ for all $n \geq 1$.

Furthermore, by (4.19), we see that if $(x_0, y_0) \in L$, then the solution (x_n, y_n) is periodic whenever θ/π is a rational number and unbounded otherwise.

Assume now that the solution (x_n, y_n) does not start on L, this to say, $P \neq 0$. We will now distinguish the three cases: $|\lambda| > \rho$, $|\lambda| < \rho$, and $|\lambda| = \rho$.

If $|\lambda| > \rho$, then by (4.19), one immediately has $x_n \to (\lambda^2 - \alpha_2)/\beta_2$ and $y_n \to \lambda$.

Suppose now that $|\lambda| < \rho$. If (x_{n_k}, y_{n_k}) is a subsequence satisfying that $\inf_k |\cos(a + (n_k - 1)\theta)| > 0$, then one obviously has $\sigma_{n_k - 1} \to \infty$. Using the definition of σ_n, one easily gets that $\sigma_{n_k}/(\sigma_{n_k - 1})$ is bounded. Then, (x_{n_k}, y_{n_k}) is a bounded subsequence, and (4.22) shows that it is attracted by the line L.

On the other hand, if $\cos(a + (n_k - 1)\theta) \to 0$, then the left equation in (4.20) leads us to $|\sigma_{n_k}(\lambda/\rho)^{n_k}| \to 2\sin\theta > 0$. Thus, $\sigma_{n_k} \to \infty$ and, using (4.20) once more, we get $\sigma_{n_k}/\sigma_{n_k - 1} \to \infty$. Therefore, (x_{n_k}, y_{n_k}) is an unbounded subsequence.

Finally, let us suppose $\rho = |\lambda|$. If we consider the change of variables

$$\begin{aligned} \overline{x} &= \left(\beta_2 x + \alpha_2 - \rho^2\right)\frac{\lambda}{2\lambda\cos\theta - 2\rho} - (y - \lambda)\frac{\rho\lambda\cos\theta}{\lambda\cos\theta - \rho}, \\ \overline{y} &= \left(\beta_2 x + \alpha_2 - \rho^2\right)\frac{1}{2\sin\theta} - (y - \lambda)\frac{(\lambda + \rho\cos\theta)}{\sin\theta}, \end{aligned} \tag{4.23}$$

then one may deduce from (4.20) that $\overline{x_n} = \rho\lambda\sigma_{n-1}/(P+\sigma_{n-1})$, $\overline{y_n} = \rho\lambda\tau_{n-1}/(P+\sigma_{n-1})$. Therefore, one immediately gets that

$$\overline{x_n}^2 + \overline{y_n}^2 = 4\frac{(\rho\lambda)^2}{(P+\sigma_{n-1})^2}, \qquad (\overline{x_n} - \rho\lambda)^2 = P^2\frac{(\rho\lambda)^2}{(P+\sigma_{n-1})^2}, \tag{4.24}$$

which clearly shows that $(\overline{x_n}, \overline{y_n})$ lies in the conic $\overline{x}^2 + \overline{y}^2 = (4/P^2)(\overline{x} - \rho\lambda)^2$, having its focus in $(0,0)$, its directrix in the line $\overline{x} = \rho\lambda$ and eccentricity $2/P$. Further, one immediately sees that the fixed point $((\lambda^2 - \alpha_2)/\beta_2, \lambda)$ is transformed by the change of variables above in $(0,0)$ and, hence, it does not belong to the conic. □

Remark 4.5. In the case $|\lambda| < \rho$ of this last theorem, one might conjecture that every subsequence of a solution (even a nonbounded one) actually approaches the line L, but this is not the case. Let us take, for example, the system with $\alpha_1 = 1$, and $\beta_1 = 3$, $\alpha_2 = -4$, $\beta_2 = -10$, in which the characteristic roots of the associated polynomial are given by $\lambda = 1$ and $\sqrt{2}e^{i\pi/4}$ and consider the solution starting on $(x_0, y_0) = (-11/20, 3/2)$. We then have that $a = 0$, $P = 1$, and $\sigma_{2+4k} = 0$ for all $k \geq 0$. One may use (4.22) to show that all the points of the form (x_{3+4k}, y_{3+4k}) lay on the line $10x + 2y + 3 = 0$ while the line L is given by $10x + 2y + 2 = 0$. Note, however, that the subsequences (x_{4k}, y_{4k}), (x_{1+4k}, y_{1+4k}), and (x_{2+4k}, y_{2+4k}) are all bounded and converge respectively to $(-3/5, 2)$, $(-2/5, 1)$, and $(-1/5, 0)$, which do belong to L.

It should also be noticed that the fixed point lays on the line $10x + 2y + 3 = 0$. This is also the case in the general setting. It follows from (4.22) that whenever $\sigma_{n_k-1} = 0$ then the point (x_{n-k}, y_{n-k}) is on the line containing the fixed point which is parallel to L.

Remark 4.6. Notice that, according to the results in [8], when $|\lambda| = \rho$ and the argument θ of the complex root is a rational multiple of π, the system is globally periodic.

4.4. Stability of Fixed Points

We finish this section with the complete study of the stability of the fixed points in the case $\beta_2 \neq 0$.

Theorem 4.7. *Suppose that $\beta_2 \neq 0$, let λ be a real eigenvalue of the matrix A given in* (2.2)*. Let $((\lambda^2 - \alpha_2)/\beta_2, \lambda)$ be the associated fixed point and denote by $\rho(A)$ the spectral radius of A.*

1. *If $|\lambda| < \rho(A)$, then the associated fixed point is unstable.*
2. *If $|\lambda| = \rho(A)$, then the associated equilibrium is stable if and only if every eigenvalue whose modulus is $\rho(A)$ is a simple eigenvalue. Moreover, the stability is asymptotic if and only if λ is a simple eigenvalue and it is the unique eigenvalue of A whose modulus is $\rho(A)$.*

Proof. The first statement was already proved in Proposition 2.4. Besides, in such proposition, we have shown that if every eigenvalue whose modulus is $\rho(A)$ is simple then the associated equilibrium is stable. Let us prove the converse.

According to the results of the previous subsections, the only cases in which one has a nonsimple eigenvalue of maximal modulus are the cases treated in Proposition 4.2(1) and (4) and the first case of Proposition 4.3. We will see that in such cases the equilibrium points associated to eigenvalues of maximal modulus are unstable.

We begin with the case of an eigenvalue λ_1 of maximal modulus with multiplicity 2. For each $N \in \mathbb{N}$, $N > 1$, one may consider the solution with initial conditions $(x_0, y_0) = ((\lambda_1^2 - \alpha_2)/\beta_2 - 2\lambda_1^2 N/((N^2+1)\beta_2), \lambda_1 - \lambda_1 N/(N^2+1))$. The solution of (4.2) in such case is given by $v_n = \lambda_1^{n+1}(N^2+1-Nn-N)/(N^2+1)$, which cannot vanish since $N > 1$. For this solution, one has $|y_N - \lambda_1| = |\lambda_1|N$, proving that the equilibrium $((\lambda_1^2-\alpha_2)/\beta_2, \lambda_1)$ is unstable.

Similarly, if A has a unique eigenvalue λ of multiplicity 3 then, for each $N \in \mathbb{N}$, $N \neq 0$ let us consider $(x_0, y_0) = ((\lambda^2-\alpha_2)/\beta_2 - 2(\lambda^2/\beta_2 N^2), \lambda)$. The corresponding solution to (4.2) is given by $v_n = ((N^2 - n - n^2)\lambda^{n+1})/N^2$. It is not difficult to see that $v_n \neq 0$ for all $n \geq 1$, and then the solution to our System (1.2) is complete. Further, since $y_n = v_n/v_{n-1}$, one gets that $|y_N - \lambda| = 2|\lambda|$. Therefore, the fixed point $((\lambda^2-\alpha_2)/\beta_2, \lambda)$ is not stable.

When $\alpha_1 = 0$, $\beta_1^2 = \alpha_2 \neq 0$, there are two equilibrium points associated to eigenvalues of maximal modulus: $(0, \pm\beta_1)$. The fixed point $(0, -\beta_1)$ is, according to the result of Proposition 4.3, unstable since the other equilibrium attracts all the solutions not starting on the line $x = 0$. To see that $(0, \beta_1)$ is also unstable, let us choose, for each odd $N \in \mathbb{N}$, the solution starting at $(x_0, y_0) = (-2\beta_1^2/N\beta_2, \beta_1)$. Then, using (4.10) and the expression for y_n given just above such equation, we have $v_n = \beta_1^{n+1} + P_1 n \beta_1^n$ if n is even and $v_n = \beta_1^{n+1} + P_1(n+1)\beta_1^n$ if n is odd, where $P_1 = -\beta_1/N$. Since N is odd, we see that y_n exists for all $n \in \mathbb{N}$ and, further, we get that $|y_N - \beta_1| = 2|\beta_1|$, which clearly implies that $(0, \beta_1)$ cannot be stable.

Finally, it only remains to prove that when A has distinct simple eigenvalues whose modulus equal $\rho(A)$, then the fixed point is not asymptotically stable, but this situation can only happen if either one has the situation described in Proposition 2.3(4) or the one given in Proposition 4.3(3). In the case of complex eigenvalues, we had seen that all the orbits lie on conics not going through the fixed point, and, hence, it cannot be asymptotically stable. In the other case, it is clear that the fixed points $(0, \pm\sqrt{\alpha_2})$ are not attracting, since every solution starting on the line $x = 0$ is 2-periodic. □

Remark 4.8. It is interesting to notice that, in the three cases in which there is an eigenvalue of maximal modulus with multiplicity larger than 1, the corresponding fixed point is attracting but unstable.

5. Nonnegative Solutions to the System with Nonnegative Coefficients

When the coefficients of our System (1.2) are nonnegative and we restrict ourselves to nonnegative initial conditions, many of the cases studied in the previous sections cannot appear. Further, in such case, one may describe which kind of orbits appear and their asymptotic behaviour without the previous calculation of the characteristic roots.

It should be noticed that whenever the coefficients in System (1.2) are nonnegative and $\alpha_1\beta_2 \neq \alpha_2\beta_1$, every initial condition (x_0, y_0) with $x_0 \geq 0$, $y_0 > 0$ gives rise to a complete orbit except for $\alpha_2 = 0$ where the condition $x_0 > 0$ is also necessary.

It will be convenient to independently study the case $\alpha_1\beta_2 = 0$. The next result is a simple summary of the results in Section 3 and Proposition 4.3, and, hence, we omit its proof.

Corollary 5.1. *Consider that the coefficients in System* (1.2) *are nonnegative and* $\alpha_1\beta_2 = 0 \neq \alpha_2\beta_1$.

(1) *If* $\beta_2 = 0$, *one has the following.*

(a) *When* $\alpha_2 \leq \beta_1^2$, *there are no nonnegative periodic orbits and all nonnegative solutions are unbounded, with the only exception of the case* $\alpha_2 = \beta_1^2$, $\alpha_1 = 0$, *which is globally 2-periodic.*

(b) *When* $\alpha_2 > \beta_1^2$, *there exists a nonattractive fixed point* $(\alpha_1/(\sqrt{\alpha_2} - \beta_1), \sqrt{\alpha_2})$ *and the whole line* $(\alpha_2 - \beta_1^2)x_0 = \alpha_1(\beta_1 + y_0)$ *of 2-periodic solutions. Every other nonnegative solution is bounded and converges to one of the 2-cycles.*

(2) *If* $\beta_2 \neq 0 = \alpha_1$*, then every nonnegative solution is bounded and the ones starting in the line* $x_0 = 0$ *are 2-periodic. Moreover,*

(a) *when* $\alpha_2 < \beta_1^2$*, there are two nonnegative fixed points:* $((\beta_1^2 - \alpha_2)/\beta_2, \beta_1)$*, which attracts all nonperiodic nonnegative solutions, and* $(0, \sqrt{\alpha_2})$*.*

(b) *When* $\alpha_2 = \beta_1^2$*, there is a unique nonnegative equilibrium* $(0, \beta_1)$ *which attracts all nonperiodic nonnegative solutions.*

(c) *When* $\alpha_2 > \beta_1^2$*, the unique nonnegative equilibrium is* $(0, \sqrt{\alpha_2})$ *which is not an attractor. Every nonnegative solution converges to one of the periodic solutions.*

The remaining cases are jointly treated in the following result. All the definitions and results on nonnegative matrices, which are used in its proof, may be found in [10, Chapter 8].

Proposition 5.2. *Suppose that System* (1.2) *has nonnegative coefficients and that* $\alpha_1\beta_2 \neq 0$*.*

(1) *If* $\alpha_2 \neq 0$ *or* $\beta_1 \neq 0$*, then there is a unique nonnegative (actually, positive) stable equilibrium which attracts all nonnegative solutions.*

(2) *If* $\alpha_2 = \beta_1 = 0$*, the system is globally 3-periodic with a unique equilibrium.*

Proof. Let us consider A as in (2.2). A simple calculation shows that $(A + I)^2$ is positive and, therefore, A is irreducible. Then, the spectral radius $\rho(A)$ is a strictly positive simple eigenvalue of A.

If there exists another eigenvalue λ such that $|\lambda| = \rho(A)$ then, since A is nonnegative and irreducible, the eigenvalues of A should be $\lambda_{k+1} = \rho(A)e^{ik\pi/3}$ where $k = 0, 1, 2$ and, consequently, $A^3 = \rho(A)^3 I$. The direct computation of A^3 shows that this is possible if and only if $\alpha_2 = \beta_1 = 0$ and, hence, in that case, the system is 3-periodic, and the only equilibrium is the one associated to the real eigenvalue $\rho(A)$.

In the remaining cases, $\lambda_1 = \rho(A)$ is a dominant eigenvalue and, according to our results of Propositions 2.4, 4.2 and Theorem 4.4, the corresponding fixed point is stable and attracts all complete solutions except those starting on the line

$$L = \{(x, y) : \beta_2 x = (\beta_1 - \lambda_1)(y + \lambda_1)\}. \tag{5.1}$$

Since λ_1 is the largest eigenvalue of A, one has that $\det(A - \mu I) < 0$ for all $\mu > \lambda_1$. However, $\det(A - \beta_1 I) = \alpha_1\beta_2 > 0$, showing that $\beta_1 < \lambda_1$. Thus, for every $x_0 \geq 0$ and $y_0 > 0$, one obtains $\beta_2 x_0 \geq 0$ and $(\beta_1 - \lambda_1)(y_0 + \lambda_1) < 0$, which proves that $(x_0, y_0) \notin L$.

The equilibrium associated to the eigenvalue $\lambda_1 = \rho(A)$ is $((\lambda_1^2 - \alpha_2)/\beta_2, \lambda_1)$, which is positive since, as before, one sees that $\det(A - \sqrt{\alpha_2} I) = \alpha_1\beta_2 > 0$ and hence $\lambda_1 > \sqrt{\alpha_2}$. □

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